B3 - Stochastic Processes : Mid-Semester Exam

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February 24, 2024. Time : 10.00 - 12.30 PM. **Maximum points : 15**

ALL QUESTIONS CARRY 5 POINTS. ATTEMPT ANY THREE OF THEM.

2 points will be deducted if you do not write your name on the answerscript.

You are free to use any results that you have learnt in your probability courses but please cite them clearly. Provide as many details as you can.

1. Let $X_1, \ldots, X_n$ be independent sub-gaussian random variables with zero mean, unit variance and variance factor 1. Let $a_1, \ldots, a_n \in \mathbb{R}$. Prove that for every $p \in [2, \infty)$, there is an absolute constant C such that

$$(\sum_{i=1}^n a_i^2)^{p/2} \leq \mathbb{E}\left[(\sum_{i=1}^n a_i X_i)^p\right] \leq (Cp \sum_{i=1}^n a_i^2)^{p/2}.$$

2. Let G_n be the graph with vertex set $V_n = \{-n, \ldots, n\}^d \subset \mathbb{Z}^d, d \geq 1$ and edges between $x, y \in V_n$ if $\|x - y\|_1 = \sum_{i=1}^d |x_i - y_i| = 1$. Let $p \in (0, 1)$. Define $G_n(p)$ to be the random subgraph of G_n with vertex set V_n and edges in G_n are retained independently with probability p (and deleted with probability $1 - p$). Let $I_n(p)$ be the number of isolated vertices in $G_n(p)$. Show that as $n \to \infty$,

$$(2n)^{-d} I(n, p) \to (1 - p)^{2d}, \text{ a.s..}$$

3. (a) Show that the Poisson random variable is sub-exponential.

   (b) Suppose that $X_1, X_2, \ldots$ are a sequence of zero mean sub-Gaussian random variables (not necessarily independent) with variance factor ν. Show that for an absolute constant C,

$$\mathbb{E}\left[\max_i \frac{|X_i|}{\sqrt{1 + \log i}}\right] \leq C\sqrt{\nu}.$$

4. Let $X_n, Y_n, n \geq 1$ be a sequence of $\overset{+}{-}1$ random variables (not necessarily independent). Show the following:

   (a) If $\liminf \mathbb{E}[X_n Y_n] \geq c$ then $\liminf \mathbb{P}\{X_n = Y_n\} \geq \frac{1+c}{2}$.

   (b) If $\limsup \mathbb{E}[X_n Y_n] \leq c$ then $\liminf \mathbb{P}\{X_n \neq Y_n\} \geq \frac{1-c}{2}$.

5. If $M_t, t \geq 0$ is a martingale and ϕ is a convex function such that $\mathbb{E}[|\phi(M_t)|] < \infty$ for all t, then $\phi(M_t), t \geq 0$ is a sub-martingale. If M_t is a sub-Martingale and ϕ is an increasing convex function such that $\mathbb{E}[|\phi(M_t)|] < \infty$ for all t, then $\phi(M_t), t \geq 0$ is a sub-martingale.

Possible Hints-

Sub-Gaussian random variable with variance factor ν: $\Psi_{X-\mu}(s) \leq s^2\nu/2$, for $s \in \mathbb{R}$ and where Ψ is the cumulant generating function i.e., $\Psi_X(s) = \log \mathbb{E}[e^{sX}]$.

Sub-exponential random variable with parameters ν and α: $\Psi_{X-\mu}(s) \leq s^2\nu/2$, for $|s| \leq \alpha^{-1}$.

Hölder's inequality: Let $w_j \geq 0$, $\sum_{j=1}^n w_j \leq 1$ be weights and Y_j's be non-negative random variables such that $\mathbb{E}\left[Y_j^{1/w_j}\right] < \infty$ for all j. Then, we have that

$$\mathbb{E}\left[\prod_{j=1}^n Y_j\right] \leq \prod_{j=1}^n \mathbb{E}\left[Y_j^{1/w_j}\right]^{w_j}.$$

Gamma function: $\Gamma(a) = \int_0^\infty e^{-s} s^{a-1}\, ds$ for $a > 0$ and $\Gamma(a) \leq a^a$.

Stirling's Approximation: $\sqrt{2\pi n}\, (\frac{n}{e})^n\, e^{1/(12n+1)} \leq n! \leq \sqrt{2\pi n}\, (\frac{n}{e})^n\, e^{1/(12n)}$.